

**Solutions to Some of the Practice Problems for Exam 1**

I. Well-formed formulas and grammatical trees

For each of the expressions below, say whether (a) it is a wff of PL, (b) it is an abbreviation of a wff of PL, or (c) it is neither of these. If (a), provide a grammatical tree. If (b), give its unabbreviated form and give a tree for that. If (c), say *briefly* why it is not a wff.

1.  $\sim\sim(P \rightarrow T_{73})$ . It is a wff.

$$\begin{array}{c} \sim\sim(P \rightarrow T_{73}) \\ |2 \\ \sim(P \rightarrow T_{73}) \\ |2 \\ (P \rightarrow T_{73}) \\ /3 \backslash \\ P \quad T_{73} \end{array}$$

2.  $(\sim M_{21} \rightarrow Z)$  It is not a wff, because  $M_{21}$  is not a symbol of the language of PL.

3.  $Q \vdash_{PL} Q$ . It is not a wff because " $\vdash_{PL}$ " is not a symbol of the language of PL (it is a symbol of our metalanguage).

4.  $(S \rightarrow Q) \wedge (T \rightarrow \sim P)$  It is an abbreviated wff.  
 $(S \rightarrow Q) \wedge (T \rightarrow \sim P) \Rightarrow \sim((S \rightarrow Q) \rightarrow \sim(T \rightarrow \sim P))$

$$\begin{array}{c} \sim((S \rightarrow Q) \rightarrow \sim(T \rightarrow \sim P)) \\ |2 \\ ((S \rightarrow Q) \rightarrow \sim(T \rightarrow \sim P)) \\ /3 \backslash \\ (S \rightarrow Q) \quad (T \rightarrow \sim P) \\ /3 \backslash \quad /3 \backslash \\ S \quad Q \quad T \quad \sim P \\ |2 \\ P \end{array}$$

5.  $(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$  It is not a wff. The Greek letters are not symbols in the language of PL

6.  $P \in \{P, Q\}$  It is not a wff. The  $\in$  (epsilon) is not a symbol of PL.

## II. Validity and Semantic Consequence

7. Prove that for all wffs  $\phi$  and  $\psi$ ,  $\models_{PL} (\sim\phi \rightarrow \sim\psi) \rightarrow (\psi \rightarrow \phi)$   
 (This theorem is part of the proof of the Soundness Theorem. A truth-table may appear as part of your proof, but you must explain why it helps show the above theorem. This theorem)

*Proof* Let  $\phi$  and  $\psi$  be (arbitrary) wffs. We wish to show that  $\models_{PL} (\sim\phi \rightarrow \sim\psi) \rightarrow (\psi \rightarrow \phi)$ . That is, we wish to show that for all assignments  $A$ ,  $(\sim\phi \rightarrow \sim\psi) \rightarrow (\psi \rightarrow \phi)$  is true in  $A$ , that is, that  $V_A((\sim\phi \rightarrow \sim\psi) \rightarrow (\psi \rightarrow \phi)) = 1$ . Let  $A$  be an (arbitrary) assignment. Given the definition of  $V_A$ , we know that either  $V_A(\phi) = 1$  or  $V_A(\phi) = 0$ , and similarly either  $V_A(\psi) = 1$  or  $V_A(\psi) = 0$ . That gives us only four possible combinations of truth values for  $\phi$  and  $\psi$  in  $A$ , given in the table below. But these truth values for  $\phi$  and  $\psi$  determine that  $(\sim\phi \rightarrow \sim\psi) \rightarrow (\psi \rightarrow \phi)$  is true in  $A$ , as shown in the table below.

$\phi$	$\psi$	$\sim\phi$	$\sim\psi$	$(\sim\phi \rightarrow \sim\psi)$	$(\psi \rightarrow \phi)$	$(\sim\phi \rightarrow \sim\psi) \rightarrow (\psi \rightarrow \phi)$
1	1	0	0	1	1	1
1	0	0	1	1	1	1
0	1	1	0	0	0	1
1	0	1	1	1	1	1

Therefore  $\models_{PL} (\sim\phi \rightarrow \sim\psi) \rightarrow (\psi \rightarrow \phi)$ . ■

8. Prove that it is *not* the case that for all wffs  $\phi$  and  $\psi$ ,  $\models_{PL} (\phi \rightarrow \psi) \rightarrow (\sim\phi \rightarrow \sim\psi)$   
 (The most straightforward way to prove this is to show that there is an instance of the schema that is not valid. Please fully specify an assignment that shows that the instance is invalid [specify the truth-value that it assigns to every sentence letter.]

*Proof* We wish to show that it is *not* the case that for all wffs  $\phi$  and  $\psi$ ,  $\models_{PL} (\phi \rightarrow \psi) \rightarrow (\sim\phi \rightarrow \sim\psi)$ . We will show that there are wffs  $\phi$  and  $\psi$  such that it is *not* the case that  $\models_{PL} (\phi \rightarrow \psi) \rightarrow (\sim\phi \rightarrow \sim\psi)$ . It will suffice to show that  $(P \rightarrow Q) \rightarrow (\sim P \rightarrow \sim Q)$  is false in some assignment. Let  $A$  be the following assignment.

$A(P) = 0$   
 $A(Q) = 1$   
 and for all other sentence letters  $\phi$ ,  $A(\phi) = 1$ .

Then  $(P \rightarrow Q) \rightarrow (\sim P \rightarrow \sim Q)$  is false in  $A$ , as shown in the starred row below.

P	Q	$\sim P$	$\sim Q$	$(P \rightarrow Q)$	$(\sim P \rightarrow \sim Q)$	$(P \rightarrow Q) \rightarrow (\sim P \rightarrow \sim Q)$
1	1	0	0	1	1	1
1	0	0	1	0	1	1
0	1	1	0	1	0	0*
0	0	1	1	1	1	1

Therefore,  $(P \rightarrow Q) \rightarrow (\sim P \rightarrow \sim Q)$  is not a logically valid wff of PL, and therefore it is not the case that for all wffs  $\phi$  and  $\psi$ ,  $\models_{PL} (\phi \rightarrow \psi) \rightarrow (\sim \phi \rightarrow \sim \psi)$ .

9. Prove Theorem 2.12 (a)-(h) on p. 34.

10. Prove that it is *not* the case that for all sets of wffs  $\Gamma$  and  $\Delta$ , if  $\Gamma$  is satisfiable and  $\Delta$  is satisfiable, then  $\Gamma \cup \Delta$  is satisfiable.  
(Hint: The most straightforward proof specifies two satisfiable sets, and proves that no assignment can satisfy the union of them.)

*Proof* We shall prove that there are sets of wffs  $\Gamma$  and  $\Delta$ , such that  $\Gamma$  is satisfiable and  $\Delta$  is satisfiable, but  $\Gamma \cup \Delta$  is unsatisfiable. Consider the sets of wffs  $\{P\}$  and  $\{\sim P\}$ . Let A that assigns 1 to all sentence letters. Then every member of  $\{P\}$  is true in A, and so is satisfiable. Let B be an assignment such that  $A(P)=0$  and for all other sentence letters  $A(\phi)=0$ . Then every member of  $\{\sim P\}$  is true in B and so it is satisfiable. But the union of these two sets is  $\{P, \sim P\}$ . Any assignment in which one of the wffs in this set is true is one in which the other wff is false. So it is not satisfiable. ■

11. Prove that for all wffs  $\phi$ , if  $\phi$  is unsatisfiable, then  $\models_{PL} \sim \phi$ .

*Proof* Let  $\phi$  be a wff and assume that  $\phi$  is unsatisfiable. We shall show that  $\models_{PL} \sim \phi$ . From our assumption, we know that there is no assignment A such that  $\phi$  is true in A. Then in every assignment A,  $\phi$  is false in A. But then it follows that  $\sim \phi$  is true in every assignment. Therefore,  $\models_{PL} \sim \phi$ .

12. Prove that for all wffs  $\phi$  and  $\psi$ , and all sets of wffs  $\Gamma$ , if either  $\Gamma \models_{PL} \sim \phi$  or  $\Gamma \models_{PL} \psi$ , then  $\Gamma \models_{PL} (\phi \rightarrow \psi)$ .  
(Hint: Use *separation of cases*, which is the following inference rule: “From (If A then C) and (if B then C), infer (If either A or B, then C).” To apply it here, first state that you will use separation of cases. Then prove that if  $\Gamma \models_{PL} \sim \phi$ , then  $\Gamma \models_{PL} (\phi \rightarrow \psi)$ . Next prove that if  $\Gamma \models_{PL} \psi$ , then  $\Gamma \models_{PL} (\phi \rightarrow \psi)$ . Then infer, by separation of cases, that if *either*  $\Gamma \models_{PL} \sim \phi$  *or*  $\Gamma \models_{PL} \psi$ , then  $\Gamma \models_{PL} (\phi \rightarrow \psi)$ .)

*Proof* Let  $\phi$  and  $\psi$  be wffs, and let  $\Gamma$  be a set of wffs. Assume that either  $\Gamma \models_{PL} \sim \phi$  or  $\Gamma \models_{PL} \psi$ . We wish to show that  $\Gamma \models_{PL} (\phi \rightarrow \psi)$ . We use separation of cases. We first show the following two conditionals.

- (1) If  $\Gamma \models_{PL} \sim \phi$ , then  $\Gamma \models_{PL} (\phi \rightarrow \psi)$ .
- (2) If  $\Gamma \models_{PL} \psi$ , then  $\Gamma \models_{PL} (\phi \rightarrow \psi)$ .

We can then use of separation of cases to infer that if either (1) or (2), then  $\Gamma \models_{PL} (\phi \rightarrow \psi)$ .

Proof of (1): Assume  $\Gamma \models_{PL} \sim \phi$ . We wish to show that  $\Gamma \models_{PL} (\phi \rightarrow \psi)$ . That is, we wish to show that for every assignment A, if every member of  $\Gamma$  is true in A, then  $(\phi \rightarrow \psi)$  is true in A. Let A be an

assignment and assume that every member of  $\Gamma$  is true in  $A$ . We wish to show that  $(\phi \rightarrow \psi)$  is true in  $A$ . From our initial assumption, we know that any assignment in which every member  $\Gamma$  is true is one in which  $\sim\phi$  is true. So  $\sim\phi$  is true in  $A$ . But if  $\sim\phi$  is true in  $A$ , then (by definition of the valuation function),  $(\phi \rightarrow \psi)$  is true in  $A$ . So  $(\phi \rightarrow \psi)$  is true in  $A$ .

Proof of (2): Similar.

From (1) and (2), we infer by separation of cases that if either (1) or (2), then  $\Gamma \models_{PL} (\phi \rightarrow \psi)$ . ■

### III. Theorems and Syntactic Consequence

13. Prove that  $\sim\phi \vdash_{PL} (\phi \rightarrow \psi)$ . Do not use the Deduction Theorem.

*Proof* Let  $\phi$  and  $\psi$  be wffs. Then the following sequence is a derivation in PL of  $(\phi \rightarrow \psi)$  from  $\sim\phi$ .

- |    |   |  |
|----|---|--|
| 1. | $\sim\phi$  | premise  |
| 2. | $\sim\phi \rightarrow (\sim\psi \rightarrow \sim\phi)$                | Axiom, instance of AS1: $\phi = \sim\phi, \psi = \sim\psi$ |
| 3. | $(\sim\psi \rightarrow \sim\phi)$                                     | 1, 2, MP   |
| 4. | $(\sim\psi \rightarrow \sim\phi) \rightarrow (\phi \rightarrow \psi)$ | Axiom, instance of AS3: $\phi = \psi, \psi = \phi$ .       |
| 5. | $(\phi \rightarrow \psi)$   | 3, 4, MP   |

Therefore,  $\sim\phi \vdash_{PL} (\phi \rightarrow \psi)$ . ■

14. Prove that  $(R \rightarrow P) \rightarrow (T \rightarrow S) \vdash_{PL} (\sim P \rightarrow \sim R) \rightarrow (T \rightarrow S)$ . You may use the Deduction Theorem.

*Proof* We first prove that  $(R \rightarrow P) \rightarrow (T \rightarrow S), (\sim P \rightarrow \sim R), T \vdash_{PL} S$ . We will then apply the DT twice. The following is a derivation of  $S$  from  $(R \rightarrow P) \rightarrow (T \rightarrow S), (\sim P \rightarrow \sim R), T$ .

- |    |   |  |
|----|---|--|
| 1. | $(\sim P \rightarrow \sim R)$                               | premise  |
| 2. | $(\sim P \rightarrow \sim R) \rightarrow (R \rightarrow P)$ | Axiom, instance of AS3: $\phi = P, \psi = R$ . |
| 3. | $(R \rightarrow P)$   | 1, 2, MP                                       |
| 4. | $(R \rightarrow P) \rightarrow (T \rightarrow S)$           | premise  |
| 5. | $(T \rightarrow S)$   | 3, 4, MP                                       |
| 6. | $T$   | premise  |
| 7. | $S$   | 5, 6, MP                                       |

Therefore,  $(R \rightarrow P) \rightarrow (T \rightarrow S), (\sim P \rightarrow \sim R), T \vdash_{PL} S$ .

By the DT, we can infer  $(R \rightarrow P) \rightarrow (T \rightarrow S), (\sim P \rightarrow \sim R) \vdash_{PL} (T \rightarrow S)$ .

By the DT, we can infer  $(R \rightarrow P) \rightarrow (T \rightarrow S) \vdash_{PL} (\sim P \rightarrow \sim R) \rightarrow (T \rightarrow S)$ . ■

15. Prove Theorem 2.18 (a)-(f), p. 43 in the book.

16. Prove that for all wffs  $\phi$  and  $\psi$ , and all sets of wffs  $\Gamma$  and  $\Delta$ , if  $\Gamma \vdash_{PL} \phi$  and  $\Delta \vdash_{PL} (\phi \rightarrow \psi)$ , then  $\Gamma \cup \Delta \vdash_{PL} \psi$ .

*Proof* Let  $\phi$  and  $\psi$ , and  $\Gamma$  and  $\Delta$  be sets of wffs. Assume that  $\Gamma \vdash_{PL} \phi$  and  $\Delta \vdash_{PL} (\phi \rightarrow \psi)$ . We wish to show that  $\Gamma \cup \Delta \vdash_{PL} \psi$ . From our assumption, we know that there is a derivation of  $\phi$  from  $\Gamma$

and a derivation  $(\varphi \rightarrow \psi)$  from  $\Delta$ . Let  $D$  be a derivation of  $\varphi$  from  $\Gamma$  and let  $m$  be the number of wffs in it.

- 1.
- .
- .
- .
- m.  $\varphi$

Let  $E$  be a derivation  $(\varphi \rightarrow \psi)$  from  $\Delta$ , and let  $n$  be the number of wffs in it.

- 1.
- .
- .
- .
- n.  $(\varphi \rightarrow \psi)$

Now consider the sequence of wffs consisting of  $D$  followed by  $E$  followed by derivation  $(\varphi \rightarrow \psi)$ .

- 1.
- .
- .
- .
- m.  $\varphi$
- .
- .
- .
- m+n.  $(\varphi \rightarrow \psi)$
- m+n+1.  $\psi$

Wff  $m+n+1$  follows from wff  $m$  and wff  $m+n$  by MP. Let  $k$  be some other wff in this sequence. Wff  $k$  will either be an axiom, or a member of  $\Gamma$ , or a member of  $\Delta$ , or follow from previous wffs in the sequence. Therefore, every wff in the sequence is either an axiom, or a member of  $\Gamma \cup \Delta$ , or follows from previous wffs by MP. Therefore, it is a derivation of  $\psi$  from  $\Gamma \cup \Delta$ . ■

17. Prove that it is *not* the case that for all wffs  $\varphi$  and all sets of wffs  $\Gamma$ , if  $\Gamma \cup \{\varphi\}$  is inconsistent, then  $\Gamma \cup \{\sim\varphi\}$  is consistent.  
(Hint: find an instance, and prove that it is an instance.)
18. Prove that for all wffs  $\varphi$  and  $\psi$ , and sets of wffs  $\Gamma$  and  $\Delta$ , if  $\Gamma \vdash_{PL} (\sim\varphi \rightarrow \sim\psi)$  and  $\Delta \vdash_{PL} \psi$ , then  $\Gamma \cup \Delta \vdash_{PL} \varphi$ .